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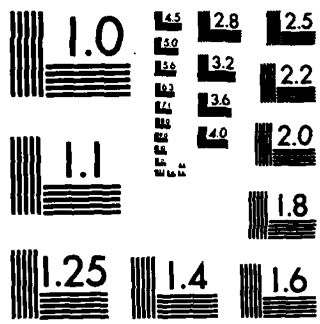

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THE GENERIC LOCAL TIME-OPTIMAL STABILIZING CONTROLS IN DIMENSION 3

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ABSTRACT

This paper studies the control system

$$\dot{x}(t) = X(x(t)) + Y(x(t))u(t), \ X(p_0) = 0, \ |u(t)| \leq 1 \ ,$$

where X and Y are C^∞ vector fields on a 3-dimensional manifold M. Under generic assumptions on X, Y, the structure of the time-optimal stabilizing controls is completely determined in a neighborhood of p_0. The proofs rely on a systematic use of a local asymptotic approximation of X and Y by means of vector fields which generate a nilpotent Lie algebra.

AMS (MOS) Subject Classifications: 49B10, 93C10

Key Words: Nonlinear control system, time optimal trajectory, asymptotic nilpotent approximation.

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SIGNIFICANCE AND EXPLANATION

Let f, g be smooth vector fields on $\mathbb{R}^d$. The problem of local stabilization for the control system

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \qquad (*)$$

with $f(0) = 0 \in \mathbb{R}^d$, $|u(t)| \leq 1$, is the following. Given a state $\bar{x}$ in a neighborhood of the origin, find a control $u(\cdot)$ that steers the system from $\bar{x}$ to the origin. If the transfer is accomplished in the shortest possible time, $u(\cdot)$ is said to be time optimal. In this paper, the time optimal local stablization problem is solved in dimension 3, under generic conditions on the nonlinear vector fields f, g. Our basic technique is a rescaling of time and space coordinates which transforms (*) into the system

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3) = (u, x_1, x_2 + kx_1^2/2) + h(x) \quad .$$

When $h \equiv 0$, an explicit solution is found. A perturbation analysis then shows that the local structure of time optimal trajectories is retained under the addition of a suitably small vector field $h(\cdot)$. As a consequence, the time optimal controls can be written in regular feedback form.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

THE GENERIC LOCAL TIME-OPTIMAL STABILIZING CONTROLS IN DIMENSION 3

Alberto Bressan*

1. Introduction

Let M be a 3-dimensional manifold, $p_0 \in M$ and let X, Y be smooth vector fields on M with $X(p_0) = 0$. Consider the control system

$$\begin{aligned} \dot{y}(t) &= X(y(t)) + Y(y(t))u(t) \\ y(0) &= p_0 \;, \end{aligned} \tag{1.1}$$

where the scalar control $u(\cdot)$ is measurable and satisfies $|u(t)| \leq 1$ almost everywhere. This paper provides a description of all admissible controls that steer the system (1.1) in minimum time from p_0 to any point p in a neighborhood of p_0. We show that the structure of the local time-optimal trajectories is completely determined by the Lie brackets up to order three of X and Y at p_0, under the generic assumptions

(A1) The vectors Y, $[Y,X]$ and $[[Y,X],X]$ are linearly independent at p_0,

(A2) $[Y,[Y,X]](p_0) = \bar{k}_1\, Y(p_0) + \bar{k}_2 [Y,X](p_0) + \bar{k}_3 [[Y,X],X](p_0)$ with $|\bar{k}_3| \neq 1$.

For the system (1.1), a numerical algorithm yielding a stabilizing control was studied in [7]. Sussmann [12] provided a complete description of time-optimal trajectories for analytic systems in the plane. ~~The present~~ This work is part of a general program of research whose goal is to determine the local properties of control systems of ~~the~~ a certain form (1.1) from the linear relations among the Lie brackets of X and Y at p_0. Our main technique is the local approximation of (1.1) by means of a nilpotent system defined on the same state space [1]. Somewhat different approximations were discussed in [3, 6] and applied in [8, 11] to obtain results on local controllability. From (1.1), a suitable rescaling of time and space coordinates leads us to the system

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$$(\dot{x}_1,\dot{x}_2,\dot{x}_3) = (u,x_1,x_2 + kx_1^2/2) + h(x) \quad ,$$
$$(x_1,x_2,x_3)(0) = (0,0,0) \quad , \quad t \in [0,1] \quad , \tag{1.2}$$

where $k = \bar{k}_3$, and the vector field $h(\cdot)$ is as small as we please, togehter with all of its high-order partial derivatives. In the special case $h \equiv 0$, the trajectories of (1.2) are easily computed as integrals of the control. The time-optimal controllability problem can therefore be explicitly solved applying Pontryagin's Maximum Principle. We use the directional convexity of the reachable set and a global necessary condition [2] to rule out the optimality of bang-bang controls with more than two switchings. In the general case, h can be regarded as a small perturbation. Repeated applications of the implicit function theorem complete the proof. The asymptotic approximation technique used here appears to be quite general and might be effective in the study of higher dimensional systems as well.

2. The Main Theorem.

As a preliminary, notice that if (A1) holds, by the implicit function theorem the equation

$$[Y,[Y,X]](y) = k_1(y)Y(y) + k_2(y)[Y,X](y) + k_3(y)[[Y,X],X](y) \tag{2.1}$$

uniquely defines the smooth functions $k_i(y)$ in a neighborhood V of p_0. If (A2) holds with $|\bar{k}_3| > 1$, we can also assume $|k_3(y)| > 1$ for all $y \in V$. Two special families of trajectories will be considered.

Definition. Let $y(\cdot)$ be an absolutely continuous map from $[0,T]$ into M with $y(0) = p_0$. We say that y is a BBB-trajectory for the system (1.1) if there exist $0 \le \tau_1 \le \tau_2 \le T$ such that

$$\dot{y} = X(y) + Y(y) \quad \text{or} \quad \dot{y} = X(y) - Y(y) \tag{2.2}$$

on each one of the (possibly empty) subintervals $(0,\tau_1)$, (τ_1,τ_2), (τ_2,T). We call $y(\cdot)$ a BSB-trajectory if there exist $0 \le \tau_1 \le \tau_2 \le T$ such that (2.2) holds on $(0,\tau_1)$ and on (τ_2,E), while

$$\dot{y} = X(y) + k_3^{-1}(y)Y(y) \tag{2.3}$$

on (τ_1,τ_2).

Our main result states that the bang-bang and the partially singular trajectories just defined are locally the only optimal ones.

Theorem 1. Consider the system (1.1) and let (A1), (A2) hold.

i) If $|\bar{k}_3| < 1$, then there exists a neighborhood V of p_0 in M such that every time-optimal trajectory steering p_0 to a point $p \in V$ is a BBB-trajectory.

ii) If $|\bar{k}_3| > 1$, then there exists a neighborhood V of p_0 such that every trajectory steering p_0 to a point $p \in V$ in minimum time is either a BBB- or a BSB-trajectory.

By inverting time and the vector fields X, Y, Theorem 1 thus yields the solution of the generic local time-optimal stablization problem in dimension three. A noteworthy consequence is that, at least for analytic X and Y, this solution can be written in regular feedback from [13]. When $|\bar{k}_3| < 1$, (1.1) behaves essentially like a linear system. Part i) in Theorem 1 could already be deduced from [10]. When $|\bar{k}_3| > 1$, the nonlinearities begin to play a major role, and a careful analysis is required. In sections 3, 4 we prove that Theorem 1 is a consequence of an analogous result (Theorem 2) concerning the system (1.2). The main steps in the proof of Theorem 2 are collected in §5. Technical details are then worked out in §§6 to 10, which may be skipped in a first reading.

3. An Equivalent Result.

By introducing a suitable set of coordinates, (1.1) will be transformed into a more tractable system on $\mathbb{R}^3$. In the following, the variable in $\mathbb{R}^3$ is $x = (x_1, x_2, x_3)$ and $\{e_1, e_2, e_3\}$ denotes the canonical orthonormal basis. Given a smooth vector field $g = (g_1, g_2, g_3)$ on $\mathbb{R}^3$, its partial derivatives are written

$$g_{i,j} = \frac{\partial g_i}{\partial x_j} \quad , \quad g_{i,jk} = \frac{\partial^2 g_i}{\partial x_j \partial x_k} \; , \; \ldots$$

∇g denotes the 3×3 matrix $(g_{i,j})$ of first order partials of g. Consider the map

$$\Theta : (s_1, s_2, s_3) \to (\exp s_1 Y) \circ (\exp s_2 [Y,X]) \circ (\exp s_3 [[Y,X],X])(p_0) \quad , \tag{3.1}$$

where $(\exp sZ)(p)$ is the value at time s of the solution of the Cauchy problem

$$\dot{y}(t) = Z(y(t)) \quad , \quad y(0) = p \in M \quad .$$

Because of (A1), Θ defines a local chart of a neighborhood of p_0. In this chart, the system (1.1) becomes

$$\dot{x} = f(x) + e_1 u \quad , \quad x(0) = 0 \in \mathbb{R}^3 \quad . \tag{3.2}$$

The vector field f can be written in the form

$$f(x) = (\bar{k}_1 x_1^2/2 \quad , \; x_1 + \bar{k}_2 x_1^2/2 \quad , \; x_2 + \bar{k}_3 x_1^2/2) + \tilde{f}(x) \tag{3.3}$$

with $\tilde{f}_{i,j}(0) = \tilde{f}_{i,11}(0) = 0$ for $i = 1,2,3,\; j = 1,2$.

Since the problem is local, we can assume that Θ is defined on some open ball $B_r \subseteq \mathbb{R}^3$ centered at the origin with radius r, and that f can be extended outside B_r to a C^∞ vector field, still called f, with compact support. We now apply to (3.2) the asymptotic rescaling procedure discussed in [1]. Consider the orthogonal decomposition $\mathbb{R}^3 = W_1 \oplus W_2 \oplus W_3$ with $W_i = \{\xi e_i ; \xi \in \mathbb{R}\}$. Let $\pi_i : \mathbb{R}^3 \to W_i$ be the canonical projections. Given an admissible control $u(\cdot)$, let $t \to x(u,t)$ be the corresponding trajectory of (3.2). If u is defined on the time-interval $[0,\varepsilon]$, construct the rescaled control $u_\varepsilon : [0,1] \to \mathbb{R}$ by setting $u_\varepsilon(t) = u(\varepsilon t)$. Moreover, set

$$x^\varepsilon(u_\varepsilon, t) = \sum_{i=1}^{3} \varepsilon^{-i} \pi_i(x(u, \varepsilon t)) \quad . \tag{3.4}$$

A direct computation shows that x^ε is the response of the system

$$\dot{x}(t) = f^\varepsilon(x(t)) + e_1 u_\varepsilon(t) \quad , \quad x(0) = 0 \in \mathbb{R}^3 \tag{3.5}$$

with $f^\varepsilon = (f_1^\varepsilon, f_2^\varepsilon, f_3^\varepsilon)$,

$$f_i^{\varepsilon}(x) = \varepsilon^{1-i} f_i\left(\sum_{j=1}^{3} \varepsilon^j \pi_j(x)\right) \quad . \tag{3.6}$$

For every $\varepsilon > 0$, (3.5) is merely a linear rescaling of (3.2). Therefore, a control u is time-optimal for (3.2) on $[0,\varepsilon]$ if and only if the corresponding u_ε is time-optimal for (3.5) on $[0,1]$. Because of (3.3), the main result proved in [1] now implies that, as $\varepsilon \to 0$, f^ε converges to the vector field

$$\bar{f}(x) = (0, x_1, x_2 + \bar{k}_3 x_1^2/2) \tag{3.7}$$

together with all partial derivatives, uniformly on bounded sets. Theorem 1 thus becomes a consequence of the following result concerning the system (1.2). If $k > 0$, we write Ω_k for the open box $(-2,2) \times (-1,1) \times (-1-k, 1+k) \subset R^3$, $C^3(\Omega_k)$ for the Banach space of three times continuously differentiable vector fields on Ω_k, and we let F be the family of all neighborhoods of the null vector field in $C^3(\Omega_k)$.

Theorem 2.

a) If $0 < k < 1$, then there exists $V \in F$ such that for all $h \in V$, $0 < T < 1$, every time-optimal control $u(\cdot)$ for (1.2) on $[0,T]$ is bang-bang with at most two switchings.

b) If $k > 1$, then there exists $V \in F$ such that, given any $h \in V$, every time-optimal control u for (1.2) on $[0,T] \subseteq [0,1]$ has the following property. Either u is bang-bang with finitely many switchings on $[0,T]$, or there exist $0 < t_1 < t_2 < T$ such that $u(t)$ is constantly equal to $+1$ or -1 on $[0,t_1]$ and on $[t_2,T]$, while $u(t) = k_3^{-1}(x(t))$ on (t_1,t_2). Here $k_3(x)$ is the third coefficient in the linear relation

$$[e_1,[e_1,g]](x) = k_1(x)e_1 + k_2(x)[e_1,g](x) + k_3(x)[[e_1,g],g](x) \quad , \tag{3.8}$$

with $g = \bar{f} + h$.

c) If $k > 1$, then there exists $V \in F$ such that, if $h \in V$ and u is a bang-bang control with initial switchings at times $0 < t_1 < t_2 < t_3 = 1$, then u is not time-optimal for (1.2) after time 1.

As usual, statements concerning controls in L^1 are always meant "up to L^1-equivalence".

4. Proof of Theorem 1.

Let Theorem 2 hold. By possibly replacing Y with $-Y$ in (A2) we can assume $\bar{k}_3 > 0$. Consider the case $0 < \bar{k}_3 < 1$ first. Set $k = \bar{k}_3$ and choose the neighborhood $V \in F$ according to a) in Theorem 2. Choose $\varepsilon > 0$ so small that the reachable set at time ε for the system (1.1) is contained within the range of the chart θ, i.e. $R(\varepsilon) \subset \theta(B_r)$, and such that $\varepsilon\Omega_k \subset B_r$, $h = f^{\varepsilon} - \bar{f} \in V$. This is possible because, as $\varepsilon \to 0$, the convergence of f^{ε} to $\bar{f}$ in (3.6), (3.7) is uniform on the bounded set Ω_k [1]. If the control u steers the system (1.1) from p_0 to some point $p \in R(\varepsilon)$ in minimum time $\eta < \varepsilon$, then the control $t \to u_{\varepsilon}(t) = u(\varepsilon t)$ is time optimal for the system (1.2) on the interval $[0, \eta\varepsilon^{-1}] \subseteq [0,1]$. By a) in Theorem 2, u_{ε} is bang-bang with at most two switchings, hence the same holds for u. Taking $V = R(\varepsilon)$, this proves i) in Theorem 1. The proof of ii) is similar. If $\bar{k}_3 > 1$, set $k = \bar{k}_3$ and choose $V \in F$ according to b) and c) in Theorem 2. Choose $\varepsilon > 0$ such that $R(\varepsilon) \subset \theta(B_r)$, $\varepsilon\Omega_k \subset B_r$, $f^{\eta} - \bar{f} \in V$ for every $\eta \in [0,\varepsilon]$. If $0 < \eta < \varepsilon$ and the control u is time-optimal for (1.1) on $[0,\eta]$, then, setting $h = f^{\varepsilon} - \bar{f}$, the control $t \to u_{\varepsilon}(t) = u(\varepsilon t)$ is optimal for (1.2) on $[0, \eta\varepsilon^{-1}] \subseteq [0,1]$. By b) in Theorem 2, either u_{ε} is partly singular, or u_{ε} is bang-bang with finitely many switchings, hence the same holds for u. In the first case, comparing (3.8) with (2.1) one concludes that u generates a BSB-trajectory, because the linear relations among the Lie brackets of the vector fields f, e_1 are preserved under the transformation (3.6). In the second case, if u has more than two switchings inside $[0,\eta]$, let $0 < t_1 < t_2 < t_3 = \eta' < \eta$ be its first three switching times. The control $t \to u_{\eta'}(t) = u(\eta' t)$ has then its third switch at $t = 1$. Since $f^{\eta'} - \bar{f} \in V$, using c) we see that $u_{\eta'}$ is not optimal after time 1, hence u is not optimal at time $\eta > \eta'$, a contradiction. Taking $V = R(\varepsilon)$, this completes the proof of part ii).

5. Sketch of the Proof of Theorem 2.

In the following, we denote $\tilde{f}(x)$ the vector field with components $(0, x_1, x_2 + kx_1^2/2)$; h is the small perturbation and $g = \tilde{f} + h$. We write B_ε for the open ball centered at the origin with radius ε. When $h \equiv 0$, the exact solution of (1.2) is

$$
\begin{aligned}
x_1(u,t) &= \int_0^t u(s)ds \\
x_2(u,t) &= \int_0^t (t-s)u(s)ds \qquad\qquad (5.1)\\
x_3(u,t) &= \frac{1}{2}\int_0^t (t-s)^2 u(s)ds + \frac{k}{2}\int_0^t \left(\int_0^s u(r)dr\right)^2 ds \quad .
\end{aligned}
$$

If u is an admissible control, i.e. if $|u(t)| \leq 1$ almost everywhere, then for $t \in [0,1]$ the trajectory $t \to x(u,t)$ is contained inside the closed box $[-1,1] \times [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{k+1}{6}, \frac{k+1}{6}]$. By a classical perturbation theorem [5], there exists a bounded neighborhood $V_0 \in F$ such that, if $h \in V_0$, every admissible trajectory for (1.2) remains inside Ω_k during the time interval $[0,1]$. The neighborhood V_0 now chosen will be kept fixed throughout. The first part of our proof will single out all solutions of the Pontryagin's equations for (1.2) on any interval $[0,T] \subseteq [0,1]$.

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3) = (u + h_1(x),\ x_1 + h_2(x),\ x_2 + \frac{k}{2}x_1^2 + h_3(x)) \quad , \qquad (5.2)_1$$

$$
\begin{aligned}
(\dot{\lambda}_1, \dot{\lambda}_2, \dot{\lambda}_3) = -(&\lambda_2 + kx_1\lambda_3 + \Sigma_{i=1}^3 h_{i,1}\lambda_i \quad , \\
&\lambda_3 + \Sigma_{i=1}^3 h_{i,2}\lambda_i \quad , \quad \Sigma_{i=1}^3 h_{i,3}\lambda_i) \quad , \qquad (5.2)_2
\end{aligned}
$$

$$(x_1,x_2,x_3)(0) = (0,0,0), \quad (\lambda_1,\lambda_2,\lambda_3)(T) = (\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3) \quad , \qquad (5.2)_3$$

$$u(t) \in \operatorname{sgn} \lambda_1(t) \qquad \text{a.e. on } [0,T] \quad , \qquad (5.2)_4$$

where $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3) \neq (0,0,0)$, $0 < T \leq 1$ and the convention $\operatorname{sgn} 0 = [-1,1]$ is used. Notice that for every data $\bar{\lambda}$ and T, $(5.2)_{1-4}$ has at least one solution. Indeed, the compactness of the reachable set $R(T)$ implies the existence of a control $\tilde{u}$ for which $x(\tilde{u},T) = \max\{\langle\bar{\lambda},x\rangle;\ x \in R(T)\}$. Such $\tilde{u}$ clearly yields a solution of (5.2). Different types of extremal controls arise, depending on the direction of $\bar{\lambda}$.

Proposition 1. There exists $V_1 \in F$ such that, if $h \in V_1$ and $\bar{\lambda}_3^2 < (12k+16)^{-2}(\bar{\lambda}_1^2 + \bar{\lambda}_2^2)$, then the solution (u,x,λ) of (5.2) is unique and the corresponding control u is bang-bang with at most one switching.

Proposition 2. For every $\varepsilon > 0$ there exists $V_2 \in F$ such that, if $h \in V_2$ and $\bar{\lambda}_3^2 > (12k+16)^{-2}(\bar{\lambda}_1^2 + \bar{\lambda}_2^2)$, then any solution (u,x,λ) of (5.2) satisfies

$$\ddot{\lambda}_1(t) \in [(1 - k \operatorname{sgn} \lambda_1(t)) + B_\varepsilon]\bar{\lambda}_3 \tag{5.3}$$

a.e. on $[0,T]$.

The two above results together imply part a) of Theorem 2. Indeed, let $0 < k < 1$ and choose the neighborhoods V_1, V_2 according to Proposition 1 and 2 with $\varepsilon = (1-k)/2$. If $h \in V_1 \cap V_2$ and if (u,x,λ) is a solution of (5.2), then either $\bar{\lambda}_3^2 < (12k+16)^{-2}(\bar{\lambda}_1^2 + \bar{\lambda}_2^2)$ and by Proposition 1 u is bang-bang with at most one switching, or $\bar{\lambda}_3^2 > (12k+16)^{-2}(\bar{\lambda}_1^2 + \bar{\lambda}_2^2)$. In this case, by (5.3) and the choice of ε, $\ddot{\lambda}_1(t)$ has a.e. the same sign of $\lambda_3(T) = \bar{\lambda}_3 \neq 0$. Hence λ_1 is either strictly concave or strictly convex on $[0,T]$ and can vanish at most at two distinct points. The corresponding control u is therefore bang-bang with no more than two switchings. Next, we assume $k > 1$ and study the case where the third component of $\bar{\lambda}$ is large compared with the others.

Proposition 3. If $k > 1$, there exists $V_3 \in F$ such that every solution (u,x,λ) of (5.2) with $h \in V_3$, $\bar{\lambda}_3^2 > (12k+16)^{-2}(\bar{\lambda}_1^2 + \bar{\lambda}_2^2)$, $\bar{\lambda}_3 < 0$, has the following property. There exist $0 < \tau_1 < \tau_2 < T$ such that u is constantly equal to $+1$ or -1 on $[0,\tau_1]$ and on $[\tau_2,T]$, while $u(t) = k_3^{-1}(x(t))$ on (τ_1,τ_2). Here $k_3(x)$ is the scalar function defined at (3.8).

Proposition 4. If $k > 1$, there exists $V_4 \in F$ such that, for every solution (u,x,λ) of (5.2) with $h \in V_4$, $\bar{\lambda}_3^2 > (12k+16)^{-2}(\bar{\lambda}_1^2 + \bar{\lambda}_2^2)$ and $\bar{\lambda}_3 > 0$, either the control u is bang-bang with finitely many switchings on $[0,T]$, or $u(t) = k_3^{-1}(x(t))$ throughout $[0,T]$.

Propositions 1, 3 and 4 clearly imply part b) of Theorem 2. To prove c), define the set of vectors

$$\Lambda = \{w = (w_1, w_2, w_3) \in \mathbb{R}^3;\ w_3^2 > (12k + 16)^2 (w_1^2 + w_2^2)\} \quad .$$

Choose $V_1 \in F$ according to Proposition 1. An application of Theorem 2 in [2] yields

Corollary 1. If $h \in V_1$, then the reachable set $R(1)$ for the system (1.2) is Λ - convex, i.e. $R(1)$ contains the point $\xi p + (1-\xi)q$ whenever $p, q \in R(1)$, $\xi \in [0,1]$ and $p-q \in \Lambda$.

Let now u be a bang-bang control satisfying Pontryagin's conditions and having a third switch at time $t = 1$. To prove that the value $x(u,1)$ of the corresponding trajectory at time 1 lies in the interior of $R(1)$, it suffices to exhibit a second admissible control. say u', such that

$$x_i(u',1) = x_i(u,1) \quad \text{for} \quad i = 1,2 \quad , \quad x_3(u',1) > x_3(u,1) \quad . \tag{5.4}$$

Indeed, if (u,x,λ) is a solution of (5.2), then $\lambda_3(1) > 0$ because of Propositions 1 to 3. The vector $w = x(u',1) - x(u,1) = (0,0,x_3(u',1) - x_3(u,1))$ therefore has a positive inner product with $\lambda(1)$ and lies in the interior of Λ. By Theorem 1 in [2], $x(u,1) \in \operatorname{int} R(1)$. To complete the proof, we only need to show that such a control u' always exists. For $a, b, c > 0$ define the control $u^+ = u^+(a,b,c)$ by setting

$$\begin{aligned} u^+(a,b,c)(t) &= 1 \quad \text{for} \quad t \in [0,a) \cup [a+b, a+b+c) \quad , \\ u^+(a,b,c)(t) &= -1 \quad \text{for} \quad t \in [a, a+b) \cup [a+b+c, \infty) \quad . \end{aligned} \tag{5.5}$$

If $\alpha,\beta,\gamma > 0$, define $u^-(\alpha,\beta,\gamma)(t) = -u^+(\alpha,\beta,\gamma)(t)$. Call $x^+ = x^+(a,b,c)$ the point reached by the system (1.2) at time $T = a+b+c$, subject to the control $u^+(a,b,c)$ and define $x^- = x^-(\alpha,\beta,\gamma)$ similarly. In the special case $h \equiv 0$, the components of x^+, x^- can be explicitly computed from (5.1):

$$x_1^+ = a-b+c \quad , \quad x_2^+ = (a+b+c)^2/2 - (b+c) + c^2 \quad ,$$

$$x_3^+ = \frac{1}{3}\{\frac{1}{2}(a+b+c)^3 - (b+c)^3 + c^3 + k[a^3 + (b-a)^3 + \frac{1}{2}(c-b+a)^3]\} \quad , \tag{5.6}$$

$$x_1^- = -\alpha+\beta-\gamma \quad , \quad x_2^- = -(\alpha+\beta+\gamma)^2/2 + (\beta+\gamma)^2 - \gamma^2 \quad ,$$

$$x_3^- = \frac{1}{3}\{-\frac{1}{2}(\alpha+\beta+\gamma)^3 + (\beta+\gamma)^3 - \gamma^3 + k[\alpha^3 + (\beta-\alpha)^3 + \frac{1}{2}(\gamma-\beta+\alpha)^3]\} \quad .$$

The three conditions

$$x_1^+ = x_1^- \quad , \quad x_2^+ = x_2^- \quad , \quad a+b+c = \alpha+\beta+\gamma = T \tag{5.7}$$

imply the relations

$$\alpha = bc/(a+c) \quad , \quad \beta = a+c \quad , \quad \gamma = ab/(a+c) \quad , \tag{5.8}$$

$$a = \beta\gamma/(\alpha+\gamma) \quad , \quad b = \alpha+\gamma \quad , \quad c = \alpha\beta/(\alpha+\gamma) \quad . \tag{5.9}$$

When these are satisfied, we have $\Delta x = x^+(a,b,c) - x^-(\alpha,\beta,\gamma) = (0,0,x_3^+ - x_3^-)$ and a direct calculation (see Appendix) shows that

$$\begin{aligned} x_3^+ - x_3^- &= [(a+b+c) - k(a-b+c)]abc/(a+c) \\ &= [(\alpha+\beta+\gamma) + k(\alpha-\beta+\gamma)]\alpha\beta\gamma/(\alpha+\gamma) \quad . \end{aligned} \tag{5.10}$$

If $a,b,c > 0$ and $u^+(a,b,c)$ satisfies the Maximum Principle on $[0, T+\varepsilon]$ for some $\varepsilon > 0$, then the corresponding adjoint variable λ in (5.2) satisfies

$$\lambda_3(t) = \overline{\lambda}_3 > 0 \qquad \forall\, t \in [0,T] \quad ,$$

$$\lambda_1(a) = \lambda_1(a+b) = \lambda_1(a+b+c) = 0 \quad ,$$

$$\ddot{\lambda}_1(t) = (1+k)\overline{\lambda}_3 \quad \text{for} \quad t \in (a, a+b) \quad ,$$

$$\ddot{\lambda}_1(t) = (1-k)\overline{\lambda}_3 \quad \text{for} \quad t \in (a+b, a+b+c) \quad .$$

The above relations imply $(k+1)b = (k-1)c$. Using this equality in (5.10) we obtain

$$x_3^+ - x_3^- = (1-k)a^2bc/(a+c) < 0 \quad . \tag{5.11}$$

If $u = u^+(a,b,c)$, consider the control $u' = u^-(\alpha,\beta,\gamma)$ with α, β, γ defined at (5.8). When $T = a+b+c = 1$, (5.7) and (5.11) imply (5.4). Therefore u cannot be optimal after time $T = 1$. The case where the bang-bang control u takes initially the value -1 can be treated similarly. Let $u = u^-(\alpha,\beta,\gamma)$ for some α, β, $\gamma > 0$. If Pontryagin's equations (5.2) are satisfied, then $(k-1)\beta = (k+1)\gamma$. Consider the control $u' = u^+(a,b,c)$

with a, b, c defined in terms of α, β, γ at (5.9). From (5.10) and the above equality we now obtain

$$x_3^+ - x_3^- = (k+1)\alpha^2\beta\gamma/(\alpha+\gamma) > 0 \quad . \tag{5.12}$$

When $T = \alpha+\beta+\gamma = 1$, (5.7) and (5.12) imply (5.4). Therefore $u = u^-$ cannot be optimal after time $T = 1$. This establishes part c) of Theorem 2 in the case $h \equiv 0$. Thanks to the implicit function theorem, the above arguments remain valid when a small perturbation h is added to the vector field $\bar{f}$ in (1.2).

Proposition 5. There exists $V_5 \in F$ such that, if $h \in V_5$ and if u is a bang-bang control with initial switchings at times $t_i : 0 < t_1 < t_2 < t_3 = 1$ which satisfies Pontryagin's equations (5.2) on $[0,1]$ with $\lambda_1(1) = 0$, then there exists a second admissible control u' such that (5.4) holds.

This will complete the proof of Theorem 2.

6. Proof of Proposition 1.

Lemma 1. Let $k > 0$, $\lambda \in \mathbb{R}^3$ with $|\lambda| = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^{1/2} = 1$. Set $\eta = (12k+16)^{-1}$ and assume $\lambda_3^2 < \eta^2(\lambda_1^2 + \lambda_2^2)$. Then at least one of the following holds

i) $|\lambda_1| > |\lambda_2| + (2k+1)|\lambda_3| + (2k+4)\eta$

ii) $|\lambda_2| > (2k+1)|\lambda_3| + (2k+4)\eta$.

Indeed, if ii) fails, since $|\lambda_3| < \eta$ we have

$$|\lambda_1| > 1 - |\lambda_2| - |\lambda_3| > 1 - [(2k+1)|\lambda_3| + (2k+4)\eta] - \eta$$

$$> (8k+10)\eta > |\lambda_2| + (2k+1)|\lambda_3| + (2k+4)\eta \quad .$$

Lemma 2. There exists a constant $M > 0$ such that every solution (u,x,λ) of $(5.2)_{1-4}$ with $|\bar{\lambda}| = 1$, $h \in V_0$, satisfies

$$M^{-1} < |\lambda(t)| < M \qquad \forall\, t \in [0,T] \quad , \tag{6.1}$$

$$|\dot{x}_i(t)| < M \quad , \quad |\dot{\lambda}_i(t)| < M \quad , \quad i = 1,2,3 \quad , \quad t \in [0,T] \quad . \tag{6.2}$$

Proof. Since V_0 is bounded in $C^3(\Omega_k)$, the operator norms of the matrices $\nabla g(x)$ of first order partial derivatives of $g = \bar{f}+h$ satisfy a uniform bound, say $|\nabla g(x)| < N$, for all $h \in V_0$, $x \in \Omega_k$.

By $(5.2)_2$, (6.1) holds with $M = e^N$. The bounds in (6.2) follows from $(5.2)_{1-2}$ and (6.1), with a possibly larger constant M.

To prove Proposition 1, it clearly suffices to consider the case $|\bar{\lambda}| = 1$. Set $\eta = (12k+16)^{-1}$ and define $\eta' = \eta/3M$, with M being the constant in (6.1), (6.2). Choose a neighborhood $V_1 \subseteq V_0$ in F such that $|h_{i,j}(x)| < \eta'$ for all $x \in \Omega_k$, $h \in V_1$, $i,j \in \{1,2,3\}$. By Lemma 1, two cases must be considered.

Case 1. Let $|\bar{\lambda}_1| > |\bar{\lambda}_2| + (2k+1)|\bar{\lambda}_3| + (2k+4)\eta$. Then for $t \in [0,T] \subseteq [0,1]$, using $(5.2)_2$ we obtain

$$|\dot{\lambda}_3(t)| < 3\eta' M = \eta \quad ,$$

$$|\lambda_3(t)| < |\bar{\lambda}_3| + \eta \quad , \tag{6.3}$$

$$|\dot{\lambda}_2(t)| < |\bar{\lambda}_3| + \eta + 3\eta' M \quad , \tag{6.4}$$

$$|\lambda_2(t)| < |\bar{\lambda}_2| + |\bar{\lambda}_3| + 2\eta \quad ,$$

$$|\dot{\lambda}_1(t)| \le |\bar{\lambda}_2| + |\bar{\lambda}_3| + 2\eta + 2k(|\bar{\lambda}_3| + \eta) + 3\eta' M \quad ,$$
$$|\lambda_1(t)| \ge |\bar{\lambda}_1| - (|\bar{\lambda}_2| + |\bar{\lambda}_3| + 2\eta) - 2k(|\bar{\lambda}_3| + \eta) - \eta \ge \eta > 0 \quad .$$

Therefore $\lambda_1(t) \neq 0$ throughout the interval $[0,T]$. From $(5.2)_4$ we deduce $u(t) = \operatorname{sgn} \lambda_1(t) = \operatorname{sgn} \bar{\lambda}_1$. The control u is thus uniquely determined and constant throughout $[0,T]$.

Case 2. Let $|\bar{\lambda}_2| \ge (2k+1)|\bar{\lambda}_3| + (2k+4)\eta$. From $(5.2)_{1-2}$, using (6.3) and (6.4) we now obtain

$$\begin{aligned} &|\lambda_2(t)| \ge |\bar{\lambda}_2| - |\bar{\lambda}_3| - 2\eta \quad , \\ &|\dot{\lambda}_1(t)| \ge (|\bar{\lambda}_2| - |\bar{\lambda}_3| - 2\eta) - 2k(|\bar{\lambda}_3| + \eta) - 3\eta' M \ge \eta > 0 \quad . \end{aligned} \tag{6.5}$$

By (6.5), $\lambda_1(\cdot)$ is a strictly monotone function, with at most one zero. By $(5.2)_4$, the corresponding control $u(\cdot)$ is bang-bang with at most one switching inside $[0,T]$. We claim that such a control u is unique, whenever $h \in \mathcal{V}_1$, for a suitably small neighborhood $\mathcal{V}_1 \in F$. To set the ideas, assume $\bar{\lambda}_2 > 0$, the case $\bar{\lambda}_2 < 0$ being entirely analogous. Define the set

$$\Gamma = \{\lambda \in \mathbb{R}^3;\ |\lambda| = 1,\ \lambda_3^2 \le \eta^2(\lambda_1^2 + \lambda_2^2),\ \lambda_2 \ge (2k+1)|\lambda_3| + (2k+4)\eta\}$$

and fix $\bar{\lambda} \in \Gamma$, $0 < T \le 1$. For $\tau \in [0,T]$ define the control $u(\tau,\cdot)$ by setting $u(\tau,t) = 1$ when $t \in [0,\tau]$, $u(\tau,t) = -1$ when $t \in (\tau,T]$, and let $x(\tau,\cdot)$, $\lambda(\tau,\cdot)$ be the solutions of $(5.2)_{1-3}$ corresponding to the control $u(\tau,\cdot)$. Since $\bar{\lambda} \in \Gamma$, we already know that any solution of $(5.2)_{1-4}$ is of the form $(u(\tau,\cdot), x(\tau,\cdot), \lambda(\tau,\cdot))$ for some $\tau \in [0,T]$. Notice that $(5.2)_4$ holds iff either $\tau = 0$ and $\lambda_1(0,0) \le 0$, or $0 < \tau < T$ and $\lambda_1(\tau,\tau) = 0$, or $\tau = T$ and $\lambda_1(T,T) \ge 0$. Uniqueness will be established by proving that

$$\frac{d}{d\tau} \lambda_1(\tau,\tau) < 0 \qquad \forall\, \tau \in [0,T] \quad . \tag{6.6}$$

When $h \equiv 0$ in (1.2), a direct calculation yields

$$\begin{aligned} &x(\tau,s) = 2\tau - s \qquad \forall\, s \in [\tau,T] \quad , \\ &\lambda_3(\tau,s) = \bar{\lambda}_3 \quad , \quad \lambda_2(\tau,s) = \bar{\lambda}_2 + (T-s)\bar{\lambda}_3 \quad , \\ &\lambda_1(\tau,t) = \bar{\lambda}_1 + \int_t^T [\bar{\lambda}_2 + (T-s)\bar{\lambda}_3 + k(2\tau - s)\bar{\lambda}_3]\, ds \quad , \\ &\frac{d}{d\tau} \lambda_1(\tau,\tau) = -\bar{\lambda}_2 - (T-\tau)\bar{\lambda}_3 - k\tau\bar{\lambda}_3 + 2k(T-\tau)\bar{\lambda}_3 \\ &\qquad \le -\bar{\lambda}_2 + (1+2k)|\bar{\lambda}_3| + k\eta \le -(k+3)\eta < 0 \quad . \end{aligned} \tag{6.7}$$

This proves (6.6) when $h \equiv 0$. To cover the general case, notice that (6.7) holds uniformly as $(\tau, T, \bar{\lambda})$ range in the compact set $\{\tau, T \in \mathbb{R};\ 0 \leq \tau \leq T \leq 1\} \times \Gamma$. Moreover, by the implicit function theorem, the total derivative of $\lambda_1(\tau, \tau)$ w.r.t. τ depends continuously on τ, T, $\bar{\lambda}$ and on the partial derivatives of order ≤ 2 of the vector field h. Therefore, if the neighborhood $\mathcal{V}_1 \in \mathcal{F}$ is suitably small, (6.6) still holds for any $h \in \mathcal{V}_1$. This completes the uniqueness proof.

7. Proof of Proposition 2.

Again it is not restrictive to assume $|\bar{\lambda}| = 1$. In this case the assumptions imply $|\bar{\lambda}_3| > (24k+32)^{-2}$. Let M be the constant in (6.1), (6.2) and choose some $\sigma > 0$ for which

$$(24k+32)^2(9+9M+10k)M\sigma < \varepsilon \quad . \tag{7.1}$$

Choose $V_2 \in F$ contained in V_0 such that

$$|h_{i,j}(x)| < \sigma \quad , \quad |h_{i,j\ell}(x)| < \sigma \quad , \quad |h_1(x)| < \sigma \tag{7.2}$$

for all $h \in V_2$, $x \in \Omega_k$, $i,j,\ell \in \{1,2,3\}$. Since the right-hand side of $(4.3)_2$ is abolutely continuous, we can differentiate $(4.3)_2$ once more:

$$\ddot{\lambda}_1 = -\mathring{\lambda}_2 - k\dot{x}_1\lambda_3 - kx_1\mathring{\lambda}_3 - \Sigma_{i=1}^3 \Sigma_{j=1}^3 h_{i,1j}(x)\dot{x}_j\lambda_i - \Sigma_{i=1}^3 h_{i,1}(x)\mathring{\lambda}_i \quad . \tag{7.3}$$

Using the bounds (6.1), (6.2), (7.1), (7.2) and the relations

$$-\mathring{\lambda}_2 - k\dot{x}_1\lambda_3 = \lambda_3 + \Sigma_{i=1}^3 h_{i,2}(x)\lambda_i - ku\lambda_3 - kh_1(x)\lambda_3 \quad , \tag{7.4}$$

$$|\mathring{\lambda}_3| = |\Sigma_{i=1}^3 h_{i,3}\lambda_i| < 3\sigma M \quad , \quad |\lambda_3(t) - \bar{\lambda}| < 3\sigma M \quad , \quad |x_1| < 2$$

we obtain

$$|\ddot{\lambda}_1(t) - (1 - ku(t))\bar{\lambda}_3| < (9+10k+9M)M\sigma < \varepsilon(24k+32)^{-2} < \varepsilon|\bar{\lambda}_3| \quad .$$

8. **Proof of Proposition 3.**

Set $\varepsilon = (k-1)/2$ and choose $\mathcal{V}' \in \mathcal{F}$ according to Proposition 2. Choose $\mathcal{V}'' \in \mathcal{F}$ so small that, whenever $h \in \mathcal{V}''$ and $g \in \bar{f}+h$, the following conditions hold at every point $x \in \Omega_k$.

i) The vectors e_1, $[e_1,g](x)$ and $[[e_1,g],g](x)$ are linearly independent.

ii) In (3.8), $k_3(x) > 1$.

Such a $\mathcal{V}''$ exists. Indeed, when $h \equiv 0$ we have $g \equiv \bar{f}$ and $[e_1,\bar{f}](x) = (0,1,kx_1)$, $[[e_1,\bar{f}],\bar{f}](x) = (0,0,1)$, $[e_1,[e_1,\bar{f}]](x) = (0,0,k)$. In this case the coefficients of the linear combination (3.8) are $k_1(x) = k_2(x) = 0$, $k_3(x) = k>1$. By continuity, the conditions i) and ii) remain valid when h ranges within a suitably small neighborhood of the null vector field in $C^3(\Omega_k)$. Now set $\mathcal{V}_3 = \mathcal{V}' \cap \mathcal{V}''$ and let (u,x,λ) be a solution of (5.2) with $\bar{\lambda}_3^{\,2} > (12k+16)^{-2}(\bar{\lambda}_1^{\,2} + \bar{\lambda}_2^{\,2})$, $\bar{\lambda}_3 < 0$. We claim that $S = \{t \in [0,T];\ \lambda_1(t) = 0\}$ is a closed interval, possibly empty. If $t_1, t_2 \in S$, let $|\lambda_1(\tau)| = \max\{|\lambda_1(t)|\cdot\ t_1 < t < t_2\}$. If $\lambda_1(\tau) \neq 0$, then $u(t) = \operatorname{sgn} \lambda_1(t)$ is constant on a neighborhood of τ, hence λ_1 is twice differentiable at τ. Since $\bar{\lambda}_3 < 0$, (5.3) and the choice of ε imply that $\operatorname{sgn} \ddot{\lambda}_1(\tau) = \operatorname{sgn} \lambda_1(\tau)$, a contradiction that proves our claim. If S is empty, Proposition 3 trivially holds by setting $\tau_1 = \tau_2 = 0$. If S contains a single point τ, set $\tau_1 = \tau_2 = \tau$. Finally, let S be a nondegenerate interval, say $[\tau_1,\tau_2]$. We need to show that $u(t) = k_3^{-1}(x(t))$ a.e. on S. The relations $\lambda_1(t) = \dot{\lambda}_1(t) = \ddot{\lambda}_1(t) = 0$ imply

$$\langle\lambda(t),\ e_1\rangle = 0\ ,$$
$$\langle-\dot{\lambda}(t),\ e_1\rangle = \langle\lambda(t),\ \nabla g(x(t))e_1\rangle = \langle\lambda(t),\ [e_1,g](x(t))\rangle = 0\ ,$$
$$\langle\ddot{\lambda}_1(t),\ e_1\rangle = -\frac{d}{dt}\langle\lambda(t),\ [e_1,g](x(t))\rangle$$
$$= \langle\lambda(t),\ \nabla g(x(t))\ [e_1 g](x(t)) - \nabla[e_1,g](x(t))\ (g(x(t)) + u(t)e_1)\rangle$$
$$= \langle\lambda(t),\ [[e_1,g],g](x(t)) - u(t)[e_1[e_1,g]](x(t))\rangle = 0\ .$$

Since $\lambda(t)$ never vanishes, for $t \in (\tau_1,\tau_2)$ the vectors e_1, $[e_1,g](x(t))$ and $[[e_1,g],g](x(t)) - u(t)[e_1,\ [e_1,g]](x(t))$, being orthogonal to $\lambda(t)$, are linearly dependent. Because of the assumption i), $u(t)$ is uniquely determined and thus coincides with $k_3^{-1}(x(t))$, defined by (3.8).

9. Proof of Proposition 4.

Some preliminary technical results are needed.

Lemma 4. Let $\tau > 0$ and let ϕ be a twice differentiable concave scalar function, with $\phi(0) = \phi(\tau) = 0$, $\dot{\phi}(0) > 0$, and let σ, m_1, m_2 be positive constants such that

$$-m_2 \le \ddot{\phi}(t) \le -m_1 < 0, \quad |\ddot{\phi}(t) - \ddot{\phi}(t')| \le \sigma|t-t'| \tag{9.1}$$

for all $t, t' \in [0,\tau]$. Then

$$|\dot{\phi}(\tau)| \ge \dot{\phi}(0) - 4\sigma(m_1 + 2m_2)m_1^{-3}\dot{\phi}^2(0) \quad . \tag{9.2}$$

Proof. The first assumption in (9.1) implies $\tau \in [2\dot{\phi}(0)/m_2, 2\dot{\phi}(0)/m_1]$. Let $a = -\ddot{\phi}(0) > 0$ and define the energy $E(t) = \dot{\phi}^2(t)/2 + a\phi(t)$. Then

$$\left|\frac{dE(t)}{dt}\right| = |\dot{\phi}(t)(\ddot{\phi}(t) + a)| \le \dot{\phi}(0)\left(1 + \frac{2m_2}{m_1}\right)\sigma t.$$

Integrating from 0 to τ we obtain

$$|E(\tau) - E(0)| \le 2\sigma(m_1 + 2m_2)m_1^{-3}\dot{\phi}^3(0) \quad . \tag{9.3}$$

This implies (9.2) because

$$|\dot{\phi}(\tau)| - |\dot{\phi}(0)| = (\dot{\phi}^2(\tau) - \dot{\phi}^2(0))(|\dot{\phi}(\tau)| + |\dot{\phi}(0)|)^{-1}$$

$$\le |E(\tau) - E(0)|\ |\dot{\phi}(0)|^{-1} \quad .$$

Lemma 5. Let $(d_n)_{n\ge1}$ be a sequence of strictly positive numbers such that $d_{n+1} \ge d_n - Cd_n^2$ for some constant $C > 1$ and all $n \ge 1$. Then $\Sigma_{n=1}^{\infty} d_n = +\infty$.

Proof. If the series converges, then $d_n \to 0$, hence $d_n \le 1/2C$ for all $n \ge N$, with N suitably large. We claim that $d_{N+n} \ge n^{-1}d_{N+1}$ for all $n \ge 1$. Indeed, if this inequality holds for some n, then

$$d_{N+n+1} \ge \min\{x - Cx^2;\ d_{N+1}/n \le x \le 1/2C\} =$$

$$= \frac{1}{n} d_{N+1} - \frac{C}{n^2} d_{N+1}^2 \ge \left(\frac{1}{n} - \frac{C}{n^2} \cdot \frac{1}{2C}\right)d_{N+1} \ge d_{N+1}/(n+1) \quad .$$

By induction, our claim holds for every $n \ge 1$, showing that the series diverges, a contradiction.

Lemma 6. Let $h \in \mathcal{V}_0$ and let $t \to (x(t),\lambda(t))$ be any local solution of the autonomous differential equation on $\mathbb{R}^6$:

$$\dot{x}(t) = g(x(t)) + e_1 \qquad \dot{\lambda}(t) = -\lambda(t)\cdot\nabla g(x(t))$$

obtained by setting $u(t) \equiv 1$ in $(5.2)_{1-2}$. There exists a constant σ' such that

$$|(d^3/dt^3)\lambda_1(t)| < \sigma'|\lambda(t)|, \quad |(d/dt)\lambda_3(t)| < \sigma'|\lambda(t)| \tag{9.4}$$

whenever $x(t) \in \Omega_k$. The smallest possible constant σ' in (9.4) approaches zero as the vector field $h = g-\bar{f}$ tends to zero in $C^3(\Omega_k)$. The same holds for the system

$$\dot{x}(t) = g(x(t)) - e_1, \quad \dot{\lambda}(t) = -\lambda(t)\cdot\nabla g(x(t)) \quad .$$

All of the above is clear because the left hand sides in (9.4) depend continuously on x, λ and on the vector field $h \in C^3(\Omega_k)$, and vanish identically when $h \equiv 0$.

Proposition 4 can now be proved. Fix $\varepsilon = (k-1)/2$, choose $V_2, V_3 \in F$ according to Proposition 2 and 3 and set $V_4 = V_2 \cap V_3$. Let $h \in V_4$ and let (u,x,λ) be a solution of (5.2) with $|\bar{\lambda}| = 1$, $\bar{\lambda}$ satisfying the assumptions made in Proposition 4. If $\lambda_1(t) = 0$ for all $t \in [0,T]$, then $(u,x,-\lambda)$ is another solution of (5.2), hence by Proposition 3 $u(t) = k_3^{-1}(x(t))$ for all t. Now assume $\lambda_1(\tau) \neq 0$ for some $\tau \in [0,T]$. Then $[\tau,T]$ contains only finitely many zeroes of λ_1. To see this, set $m_1 = (k-1-\varepsilon)\bar{\lambda}_3$, $m_2 = (k+1+\varepsilon)\bar{\lambda}_3$. Whenever $\lambda_1(t) \neq 0$, u is constantly equal to $\operatorname{sgn} \lambda_1(t)$ on a neighborhood of t, hence λ_1 is three times differentiable at t. By (5.3)

$$-m_2 < \frac{d^2}{dt^2}|\lambda_1(t)| < -m_1 < 0 \quad . \tag{9.5}$$

If λ_1 vanishes infinitely many times inside $[\tau,T]$, let τ_0 be the smallest time. Recursively, set $\tau_{n+1} = \inf\{t \in (\tau_n,T];\ \lambda_1(t) = 0\}$. By (9.5), $\dot{\lambda}_1(\tau_0) \neq 0$ and τ_0 is an isolated zero of λ_1. By induction, one easily checks that the same holds for every n, hence the sequence $(\tau_n)_{n\geq 1}$ is strictly increasing. We now apply Lemma 4 to the function $\phi(t) = |\lambda_1(\tau_n+t)|$ for each interval $[\tau_n,\tau_{n+1}]$. The second estimate in (9.1) is obtained from (9.4) and (6.1), setting $\sigma = M\sigma'$. Using (9.2) we deduce

$$|\dot{\lambda}_1(\tau_{n+1})| > |\dot{\lambda}_1(\tau_n)| - 4\sigma(m_1+2m_2)m_1^{-3}\ |\dot{\lambda}_1(\tau_n)| \quad .$$

If infinitely many τ_n were defined, by Lemma 5 $\Sigma_{n=0}^{\infty}|\dot{\lambda}_1(\tau_n)| = +\infty$. From (9.5) it follows $\tau_{n+1} - \tau_n > 2|\dot{\lambda}_1(\tau_n)|m_2^{-1}$, hence $\lim_{n\to\infty} \tau_n = +\infty$, providing a contradiction. An analogous argument shows that λ_1 can have only finitely many zeroes inside $[0,\tau]$. Hence the corresponding control u is bang-bang with finitely many switchings.

10. Proof of Proposition 5.

We restrict the analysis to the case where $u(t) = +1$ on the initial interval $[0,t_1)$. When $u(t) = -1$ on $[0,t_1)$ an entirely analogous argument applies.

Lemma 7. For every h in a suitably small neighborhood $V \in F$, there exists a unique one-parameter family of bang-bang controls $u(\xi) = u^+(a(\xi),b(\xi),c(\xi))$, $\xi \in [0,1/2]$, having a first switch at time $t = \xi$ and a third switch at $t = 1$, which satisfy Pontryagin's equations (5.2) on the time interval $[\xi,1]$ with $\lambda_1(\xi) = \lambda_1(1) = 0$.

Proof. Whenever $h \in V$ is small enough, the proofs of Propositions 1 to 3 show that the adjoint variable $\lambda(\cdot)$ in (5.2) corresponding to a bang-bang control with at least two switchings inside $[0,1]$ must satisfy

$$\lambda_3(t) > 0, \ |\ddot{\lambda}_1(t)/\lambda_3(1) - (1 - ku(t))| < (k-1)/2 \tag{10.1}$$

a.e. on $[0,1]$. To construct the one-parameter family $u(\xi)$, for a fixed $h \in C^3(\Omega_k)$, $g = \bar{f}+h$ and $\xi \in [0,1/2]$, let $u = u^+(\xi,t_2-\xi,1-t_2)$ be the control whose value is initially $+1$ and has switchings at times ξ, t_2, 1, as in (5.5). Consider the Cauchy problem on $\mathbb{R}^6$, starting at time $t = \xi$:

$$\begin{aligned} &\dot{x}(t) = g(x(t)) + e_1 u(t) \quad , \quad \dot{\lambda}(t) = -\lambda(t)\nabla g(x(t)) \quad , \\ &x(\xi) = (\exp \xi(g+e_1))(0) \quad , \quad \lambda(\xi) = (0,\nu,1) \end{aligned} \tag{10.2}$$

for some $\nu \in \mathbb{R}$. The above data determine uniquely a trajectory $t \to (x(t),\lambda(t))$. From (10.1) it is clear that the control $u = u^+(\xi,t_2-\xi,1-t_2)$ satisfies the Maximum Principle (5.2) on a neighborhood of the interval $[\xi,1]$ iff $\lambda_1(t_2) = \lambda_1(1) = 0$. We claim that for $V \in F$ suitably small, the conditions

$$\lambda_1(t_2) = \lambda_1(1) = 0 \quad , \quad \xi < t_2 < 1 \tag{10.3}$$

implicitly define t_2, ν uniquely as functions of h, ξ, for all $h \in V$, $\xi \in [0,1/2]$. Indeed, when $h \equiv 0$, the equations (10.1), (10.3) can be solved explicitly, first for ν as a function of t_2 and ξ, then for t_2 in terms of ξ:

$$\lambda(t_2) = (t_2-\xi)(-\nu-k\xi) + (t_2-\xi)^2(1-k)/2 \quad . \tag{10.4}$$

The right-hand side of (10.4) vanishes at the point $t_2 \in (\xi,1)$ iff $\nu = (t_2-\xi)(1-k)/2 - k\xi$. In this case

$$\lambda_1(1) = (1-t_2)(t_2-\xi)(1+k)/2 + (1-t_2)^2(1-k)/2 \quad , \tag{10.5}$$

hence $\lambda_1(1) = 0$ iff

$$(t_2-\xi)/(1-t_2) = (k-1)/(k+1) \quad . \tag{10.6}$$

The exact value of t_2 as a function of ξ is immediately obtained from (10.6). From (10.6) it also follows

$$(t_2-\xi) > (k-1)/4(k+1) \quad , \quad 1+\xi-2t_2 > 0 \quad , \quad 1-t_2 > 1/4 \tag{10.7}$$

for all $\xi \in [0,1/2]$. Differentiating (10.4) and (10.5) w.r.t. ν and t_2 respectively and using (10.7) we obtain

$$\frac{\partial\lambda_1(t_2)}{\partial\nu} = \xi-t_2 < -\frac{k-1}{4(k+1)} < 0 \tag{10.8}$$

$$\frac{\partial\lambda_1(1)}{\partial t_2} = (1+\xi-2t_2)(k+1)/2 + (k-1)(1-t_2) > (k-1)/4 > 0 \quad . \tag{10.9}$$

By the implicit function theorem, there exists a neighborhood $\mathcal{V} \in \mathcal{F}$ such that (10.2), (10.3) determine (t_2,ν) uniquely as C^3 functions of (h,ξ) in $\mathcal{V} \times [0,1/2]$. This proves Lemma 7, by setting $a(\xi) = \xi$, $b(\xi) = t_2(\xi)-\xi$, $c(\xi) = 1-t_2(\xi)$.

Next, it will be shown that Proposition 5 holds if the bang-bang control u belongs to the one-parameter family $u^+(a(\xi), b(\varepsilon), c(\xi))$ just defined. To this purpose we need a technical result, whose proof is straightforward.

Lemma 8. Let $\mathcal{V} \in \mathcal{F}$ and let $(h,\xi) \to \phi(h,\xi)$ be a C^2 map from $\mathcal{V} \times [0,1/2]$ into $\mathbb{R}$ such that $\phi(h,0) = 0$ for all $h \in \mathcal{V}$ and $\phi(0,\xi) > 0$ for all $\xi \in (0,1/2]$. Assume that either i) $(\partial\phi/\partial\xi)(0,0) > 0$ or ii) $(\partial\phi/\partial\xi)(h,0) = 0$ for all $h \in \mathcal{V}$ and $(\partial^2\phi/\partial\xi^2)(0,0) > 0$. Then $\phi(h,\xi) > 0$ for all $\xi \in (0,1/2]$ and all h in some neighborhood of the null vector field in $C^3(\Omega_k)$.

For $h \in \mathcal{V}$ suitably small, we now construct a second one-parameter family of bang-bang controls $u'(\xi) = u^-(\alpha(\xi),\beta(\xi),\gamma(\xi))$, choosing α, β, γ such that $\alpha+\beta+\gamma = 1$ and the equalities in (5.4) hold, i.e.

$$\begin{aligned}&\pi_i(\exp \gamma(\xi)(g-e_1))(\exp \beta(\xi)(g+e_1))(\exp \alpha(\xi)(g-e_1))(0)\\ &= \pi_i(\exp c(\xi)(g+e_1))(\exp b(\xi)(g-e_1))(\exp a(\xi)(g+e_1))(0)\end{aligned} \tag{10.10}$$

for $i = 1,2$. When $h \equiv 0$, (10.6) implies

$$a(\xi) = \xi,\ b(\xi) = (k-1)(1-\xi)/2k,\ c(\xi) = (k+1)(1-\xi)/2k \tag{10.11}$$

and $\alpha(\xi)$, $\beta(\xi)$, $\gamma(\xi)$ are obtained substituting the values (10.11) in (5.8). By the implicit function theorem, the condition $\alpha(\xi) + \beta(\xi) + \gamma(\xi) = 1$ together with (10.10) defines a C^3 map $(h,\xi) \to (\alpha,\beta,\gamma)$ on $V \times [0,1/2]$, for a suitably small neighborhood $V \in F$. Notice that when $h \equiv 0$ and ξ ranges inside $[0,1/2]$, $\alpha(\xi)$ and $\beta(\xi)$ are strictly positive, while $\gamma(\xi) > 0$ for $\xi > 0$. Moreover, $(d\gamma/d\xi) = (k-1)/(k+1) > 0$ at $\xi = 0$. Setting $\phi = \gamma$ in Lemma 8, we deduce $\gamma(\xi) > 0$ for all $(h,\xi) \in V \times [0,1/2]$ with V small enough. Therefore the bang-bang control $u(\xi) = u^-(\alpha(\xi),\beta(\xi),\gamma(\xi))$ is well defined. To prove the last inequality in (5.4), set $\phi(h,\xi) = x_3(u'(\xi),1) - x_3(u(\xi),1)$. For any fixed h, when $\xi = 0$ (10.10) has the obvious solution $\alpha(0) = b(0)$, $\beta(0) = c(0)$, $\gamma(0) = a(0) = 0$. Call $\tilde{u}$ the control $u^+(a(0),b(0),c(0))$, which coincides with $u^-(\alpha(0), \beta(0), \gamma(0))$ for all $t \in [0,1]$, and let $t \to (\tilde{x}(t),\tilde{\lambda}(t))$ be the corresponding trajectory and adjoint variable in (10.2). Since $\tilde{\lambda}_1$ vanishes at times 0, $t_2 = b(0)$, 1, as $\xi \to 0$ we have

$$\langle\tilde{\lambda}(1),\ x(u^+(a(\xi),b(\xi),c(\xi)),1) - \tilde{x}(1)\rangle\ \xi^{-1}$$

$$= [\textstyle\int_0^1 \tilde{\lambda}_1(t)[u^+(a(\xi),b(\xi),c(\xi))(t) - u^+(a(0),b(0),c(0))(t)]dt$$

$$+ O(\xi^2)]\xi^{-1} = o(\xi) \quad .$$

The same holds for $u^-(\alpha(\xi),\beta(\xi),\gamma(\xi))$, therefore

$$\lim_{\xi\to 0} \langle\tilde{\lambda}(1),x(u^-(\xi),1) - x(u^+(\xi),1)\rangle\xi^{-1} = \tilde{\lambda}_3(1)(\partial\phi/\partial\xi)(h,0) = 0 \quad . \tag{10.12}$$

From (10.12) we deduce $(\partial\phi/\partial\xi)(h,0) = 0$. When $h \equiv 0$, (5.10) and (10.11) imply

$$\phi(0,\xi) = \frac{(k-1)^2(k+1)(1-\xi)^2}{2k[2k\xi+(k+1)(1-\xi)]} ,$$

hence $(\partial^2\phi/\partial\xi^2)(0,0) = (k-1)^2/k > 0$. By Lemma 8, $x_3(u'(\xi),1) - x_3(u(\xi),1) > 0$ for all $\xi \in (0,1/2]$ and h in a neighborhood of the null vector field.

To conclude the proof of Proposition 5, notice that for every constant $\varepsilon' > 0$, in (5.3) we can choose $\varepsilon > 0$ so small that the conditions

$$|\ddot{\lambda}_1(t)/\bar{\lambda}_3 - (1-k)| < \varepsilon \quad \text{for} \quad t \in (0,t_1) \cup (t_2,1) \ ,$$

$$|\ddot{\lambda}_1(t)/\bar{\lambda}_3 - (1+k)| < \varepsilon \quad \text{for} \quad t \in (t_1,t_2)$$

together with $\lambda_1(t_1) = \lambda_1(t_2) = \lambda_1(1) = 0$, $\lambda_1(t) > 0$ on $(0,t_1)$ imply

$$|(t_2-t_1)/(1-t_2) - (k-1)/(k+1)| < \varepsilon' \quad , \quad t_1 < (1-t_2) + \varepsilon' \quad . \tag{10.13}$$

For $\varepsilon' > 0$ suitably small, (10.13) implies $t_1 \in [0,1/2]$. Therefore, if $h \in V$ is small enough, a bang-bang control u, which is initially positive and has switchings at times $0 < t_1 < t_2 < t_3 = 1$, can satisfy Pontryagin's equations (5.2) only if $t_1 \leq 1/2$. But in this case u is the member of the one-parameter family of control functions $u^+(a(\xi),b(\xi),c(\xi))$ obtained by setting $\xi = t_1$. Hence Proposition 5 holds for u.

Appendix.

The equalities (5.10) are obtained from (5.6) to (5.9), using the relations $ab = \beta\gamma$, $\alpha\beta = bc$, as follows.

$$
\begin{aligned}
3(x_3^+ - x_3^-) &= (a+b+c)^3 - (b+c)^3 + c^3 - (\beta+\gamma)^3 + \gamma^3 \\
&\qquad + k[a^3 + (b-a)^3 - \alpha^3 - (\beta-\alpha)^3 + (a-b+c)^3] \\
&= a^3 + 3a^2(b+c) + 3a(b+c)^2 + (b+c)^3 - (b+c)^3 + c^3 - \beta^3 - 3\beta^2\gamma \\
&\quad - 3\beta\gamma^2 - \gamma^3 + \gamma^3 + k[a^3 + (b-a)^3 - \alpha^3 - \beta^3 + 3\beta^2\alpha - 3\beta\alpha^2 \\
&\quad + \alpha^3 + c^3 - 3c^2(b-a) + 3c(b-a)^2 - (b-a)^3] \\
&= a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc + 3ac^2 + c^3 - (a^3 + 3a^2c \\
&\quad + 3ac^2 + c^3) - 3a^2b^2/(a+c) - 3(a^2b + abc) + k[a^3 - (a^3 \\
&\quad + 3a^2c + 3ac^2 + c^3) + 3(abc + bc^2) - 3b^2c^2/(a+c) \\
&\quad + c^3 - 3bc^2 + 3ac^2 + 3b^2c - 6abc + 3a^2c \\
&= 3ab^2 + 3abc - 3a^2b^2/(a+c) + k[-3abc - 3b^2c^2/(a+c) + 3b^2c] \ .
\end{aligned}
$$

$$
\begin{aligned}
x_3^+ - x_3^- &= (a^2b^2 + ab^2c + a^2bc + abc^2 - a^2b^2)/(a+c) \\
&\qquad - k(a^2bc + abc^2 + b^2c^2 - ab^2c - b^2c^2)/(a+c) \\
&= \frac{abc}{a+c}[a+b+c - k(a-b+c)] = \frac{2\beta\gamma}{2+\gamma}[\alpha+\beta+\gamma+k(\alpha-\beta+\gamma)] \ .
\end{aligned}
$$

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This paper studies the control system

$$\dot{x}(t) = X(x(t)) + Y(x(t))u(t), \quad X(p_0) = 0, \quad |u(t)| \le 1 ,$$

where X and Y are C^∞ vector fields on a 3-dimensional manifold M. Under generic assumptions on X, Y, the structure of the time-optimal stabilizing controls is completely determined in a neighborhood of p_0. The proofs rely on a systematic use of a local asymptotic approximation of X and Y by means of vector fields which generate a nilpotent Lie algebra.

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